

On a pricing problem for a multi-asset option with general transaction costs

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Abstract

We consider a Black-Scholes type equation arising on a pricing model for a multi-asset option with general transaction costs. The pioneering work of Leland is thus extended in two different ways: on the one hand, the problem is multi-dimensional since it involves different underlying assets; on the other hand, the transaction costs are not assumed to be constant (i.e. a fixed proportion of the traded quantity). Using an iterative method, we prove the existence of solutions for the corresponding initial-boundary value problem. Moreover, we develop a numerical scheme that allows to find a sequence of approximate solutions. We apply this method on a specific multi-asset derivative and we obtain the option price under different pricing scenarios.

1 Introduction

The Black-Scholes model [2] relies on following different assumptions such as constant values of volatility and interest rates, the non-existence of dividend yields, the efficiency of the markets and the non-existence of transaction costs, among others. Following Leland's approach [11], transaction costs can be included on the pricing methodology by applying a discrete-time replicating strategy. A nonlinear partial differential equation is obtained for the option price, which is denoted by $V(S, t)$; namely,

$$\frac{\partial V}{\partial t} + \frac{1}{2} \hat{\sigma} \left(S \frac{\partial^2 V}{\partial S^2} \right)^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0, \quad (1)$$

where $\hat{\sigma}$ is defined based upon the transaction cost function. For example, if transaction costs are defined by a constant rate C_0 , then $\hat{\sigma}$ is given by

$$\hat{\sigma} \left(S \frac{\partial^2 V}{\partial S^2} \right)^2 = \sigma^2 \left(1 - \text{Le} \, \text{sgn} \left(S \frac{\partial^2 V}{\partial S^2} \right) \right) = \begin{cases} \sigma^2 (1 - \text{Le}) & \text{if } \frac{\partial^2 V}{\partial S^2} > 0 \\ \sigma^2 (1 + \text{Le}) & \text{if } \frac{\partial^2 V}{\partial S^2} < 0 \end{cases}$$

where $\text{Le} = \sqrt{\frac{2}{\pi}} \frac{C_0}{\sigma \sqrt{\Delta t}}$ is the Leland number. This approach was then continued and improved by [3] and [6].

Different choices of transaction costs functions lead to variations on the nonlinear term of the partial differential equation. In [1], the authors propose a non-increasing linear function and find solutions for the stationary problem. In [15], the concept of transaction costs function is generalized and the so-called *mean value modification of the transaction costs function* is developed. This transformation allows the authors to formulate a general one-dimensional Black-Scholes equation by solving the equivalent quasilinear Gamma equation. Other authors [14, 4] also find solutions to the problem with constant transaction costs and relaxing the assumptions of constant volatility and interest rate.

The main distinctive aspect in the above-cited works is that they all consider only one asset within the partial differential equation. In [17] and [18], the author generalizes the Leland approach to cover different types of multi-asset options, developing the nonlinear partial differential equation and solving numerically a list of examples.

In this work, we obtain a weak solution for the problem of pricing a multi-asset option with a general transaction cost function. We derive the following nonlinear problem

$$\begin{aligned}
-V_\tau + \mathcal{L}V &= FV \quad \text{in } \Omega \times [0, T] \\
V(x_1, \dots, x_n, 0) &= V_0(x_1, \dots, x_n) \quad \text{in } \Omega \\
V(x_1, \dots, x_n, \tau) &= g(x_1, \dots, x_n, \tau) \quad \text{in } \partial\Omega \times (0, T).
\end{aligned} \tag{2}$$

where V is the option price, \mathcal{L} is an elliptic operator and F is a nonlinear term. After transforming the problem into a one with zero initial-boundary conditions, we apply an iterative method on the linearized problem to find the solution. The iterative linear problem is set as

$$\begin{aligned}
-U_\tau^n + \mathcal{L}U^n &= F(U^{n-1} + \Lambda) \quad \text{in } \Omega \times [0, T] \\
U^n(x_1, \dots, x_n, 0) &= 0 \quad \text{in } \Omega \\
U^n(x_1, \dots, x_n, \tau) &= 0 \quad \text{in } \partial\Omega \times (0, T).
\end{aligned} \tag{3}$$

where n represents the iteration step, Λ is the solution to the Black-Scholes equation with nonzero initial and boundary conditions and \mathcal{L} is the same elliptic operator. A weak solution is found by partitioning the interval $[0, T]$ accordingly and verifying that the iteration converges on each subinterval to a solution $U \in W_\infty^{2,1}$ in the weak sense.

In the second part of the work, we develop a numerical approach in order to find a strong solution using the iterative method. For this purpose, the Alternating Difference Implicit (ADI) scheme is selected within the family of splitting operators. Different works [7, 8, 13, 12] study the applicability of this approach to deal with the mixed derivatives terms of the discretization. On multidimensional problems, the ADI method allows to solve efficiently the PDE problem by applying a tridiagonal matrix algorithm in comparison to the classical Crank-Nicholson scheme.

The structure of the paper is as follows. In Section 2 we derive the nonlinear PDE that explains the dynamics of the option price for a multi-asset derivative considering a general transaction costs function. In Section 3 we apply all the necessary steps to prove the existence of a weak solution for the transformed problem by applying an iterative method. Finally, in Section 4 we develop the ADI framework to find a strong solution and price a specific multi-asset derivative.

2 PDE derivation for multiple assets and general transaction cost function

Let Π be the portfolio that contains δ_i amount of assets S_i and an option V over those assets at time t . This portfolio can be represented by

$$\Pi = V + \sum_{i=1}^N \delta_i S_i. \tag{4}$$

Applying the Itô's formula over V , we get

$$\Delta V = \frac{\partial V}{\partial t} \Delta t + \sum_{i=1}^N \frac{\partial V}{\partial S_i} \Delta S_i + \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N \sigma_i \sigma_j \rho_{ij} S_i S_j \frac{\partial^2 V}{\partial S_i \partial S_j} \Delta t. \tag{5}$$

Transaction costs appear when calculating $\Delta \Pi$, which expresses the variation of the portfolio at each time t . Specifically, the variation of the portfolio is represented by

$$\Delta \Pi = \Delta \left(V + \sum_{i=1}^N \delta_i S_i \right) + \sum_{i=1}^N \delta_i \Delta TC_i, \tag{6}$$

where $\delta_i \Delta TC_i$ is the amount of transaction costs when buying or selling δ_i assets of S_i . By taking $\delta_i = -V_{S_i}$, we obtain

$$\Delta \Pi = \Delta V - \sum_{i=1}^N \frac{\partial V}{\partial S_i} \Delta S_i - \sum_{i=1}^N \Delta TC_i. \tag{7}$$

Following the approach in [15], it is seen that

$$\Delta TC_i = S_i C(|\Delta \delta_i|) |\Delta \delta_i|, \tag{8}$$

where C is the transaction cost function. By defining r_{TC}^i to be the expected value of the change of the transaction costs per unit time interval Δt and price S_i , we see that

$$r_{TC}^i = \frac{E[\Delta TC_i]}{S_i \Delta t} = \frac{E[C(|\Delta \delta_i|)|\Delta \delta_i|]}{\Delta t}.$$

Thus, we approximate the transaction costs by the expected value of the transaction costs function applied to the amount of assets bought or sold and multiplied by these amount again. This value is then multiplied by the price of asset S_i in order to get a transaction cost in dollar terms.

Applying (8) in (7) and using ΔV , we obtain

$$\Delta \Pi = \left(\frac{\partial V}{\partial t} + \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N \sigma_i \sigma_j \rho_{ij} S_i S_j \frac{\partial^2 V}{\partial S_i \partial S_j} \right) \Delta t - \sum_{i=1}^N S_i r_{TC}^i \Delta t. \quad (9)$$

From the assumption $\Delta \Pi = r \Pi \Delta t$ and (9), we obtain

$$rV + \sum_{i=1}^N r_{TC}^i S_i = \frac{\partial V}{\partial t} + \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N \sigma_i \sigma_j \rho_{ij} S_i S_j \frac{\partial^2 V}{\partial S_i \partial S_j} + r \sum_{i=1}^N \frac{\partial V}{\partial S_i} S_i \quad (10)$$

where $r_{TC}^i S_i = \frac{E[\Delta TC_i]}{\Delta t} = \frac{E[C(|\Delta \delta_i|)|\Delta \delta_i| S_i]}{\Delta t}$.

Equation (10) is the nonlinear PDE that represents the behaviour of the option price for a multi-asset option when defining a general transaction costs function. In order to get the complete expression of the PDE, we have to calculate δ_i . From previous steps we know that

$$\Delta \delta_i = -\Delta \frac{\partial V}{\partial S_i} \sim \sum_{j=1}^N \frac{\partial^2 V}{\partial S_i \partial S_j} \Delta S_j$$

taking only the terms with order $\Delta t^{1/2}$. Noting that

$$\Delta S_j \sim \sigma_j S_j \phi_j \sqrt{\Delta t},$$

with ϕ_j being a standard normal variable, we find that

$$|\Delta \delta_i| = \left| \sum_{j=1}^N \frac{\partial^2 V}{\partial S_i \partial S_j} \Delta S_j \right| = \left| \sum_{j=1}^N \frac{\partial^2 V}{\partial S_i \partial S_j} \sqrt{\Delta t} \sigma_j S_j \phi_j \right| = \sqrt{\Delta t} \left| \sum_{j=1}^N \frac{\partial^2 V}{\partial S_i \partial S_j} \sigma_j S_j \phi_j \right|.$$

Setting $\Phi_i = \sum_{j=1}^N \frac{\partial^2 V}{\partial S_i \partial S_j} \sigma_j S_j \phi_j$, we obtain that $\Phi_i \sim N(0, \Theta_i)$ with

$$\Theta_i = \left| \sum_{j=1}^N \sum_{k=1}^N \frac{\partial^2 V}{\partial S_i \partial S_j} \frac{\partial^2 V}{\partial S_i \partial S_k} \sigma_j \sigma_k \rho_{jk} S_j S_k \right|.$$

Therefore,

$$r_{TC}^i S_i = \frac{E[\Delta TC]}{\Delta t} = \frac{E[C(|\Delta \delta_i|)|\Delta \delta_i| S_i]}{\Delta t} = \frac{\sqrt{\Delta t} E \left[C \left(\sqrt{\Delta t} |\Phi_i| \right) |\Phi_i| S_i \right]}{\Delta t} = \frac{S_i}{\sqrt{\Delta t}} E \left[C \left(\sqrt{\Delta t} |\Phi_i| \right) |\Phi_i| \right]. \quad (11)$$

Using (11) in (10), we find the following nonlinear PDE which models the dynamic of a multi-asset option.

$$rV + \sum_{i=1}^N \frac{S_i}{\sqrt{\Delta t}} E \left[C \left(\sqrt{\Delta t} |\Phi_i| \right) |\Phi_i| \right] = \frac{\partial V}{\partial t} + \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N \sigma_i \sigma_j \rho_{ij} S_i S_j \frac{\partial^2 V}{\partial S_i \partial S_j} + r \sum_{i=1}^N \frac{\partial V}{\partial S_i} S_i. \quad (12)$$

3 Existence of solution for the resulting PDE

3.1 Setting the Nonlinear PDE

Let C be a measurable bounded transaction cost function such that $C : \mathbb{R}_0^+ \rightarrow \mathbb{R}_0^+$, $C \in L^2(\mathbb{R}_0^+)$ and let $\overline{C}, \underline{C} > 0$ be such that $\underline{C} < C(x) < \overline{C}$ for every $x \in \mathbb{R}_0^+$.

Following equation (12), we can see that the nonlinear term directly depends on the shape of the transaction costs function and the size of the discrete delta-hedging time step. In order to get a more simplified equation, we propose the change of variables $x_i = \log(S_i)$ y $\tau = T - t$, obtaining

$$\begin{aligned}\frac{\partial V}{\partial S_i} &= \frac{\partial V}{\partial x_i} \frac{1}{S_i}, \\ \frac{\partial^2 V}{\partial S_i^2} &= \frac{\partial}{\partial S_i} \left(\frac{\partial V}{\partial x_i} \frac{1}{S_i} \right) = \frac{1}{S_i^2} \left(\frac{\partial^2 V}{\partial x_i^2} - \frac{\partial V}{\partial x_i} \right), \\ \frac{\partial^2 V}{\partial S_i \partial S_j} &= \frac{\partial}{\partial S_j} \left(\frac{\partial V}{\partial x_i} \frac{1}{S_i} \right) = \frac{\partial^2 V}{\partial x_i \partial x_j} \frac{1}{S_i} \frac{1}{S_j}.\end{aligned}$$

By applying this change of variables in (12), the nonlinear PDE results on

$$rV + \sum_{i=1}^N \frac{e^{x_i}}{\sqrt{\Delta t}} E \left[C \left(\sqrt{\Delta t} |\Phi_i| \right) |\Phi_i| \right] = -\frac{\partial V}{\partial \tau} + \frac{1}{2} \sum_{i=1}^N \sigma_i^2 \left(\frac{\partial^2 V}{\partial x_i^2} - \frac{\partial V}{\partial x_i} \right) + \frac{1}{2} \sum_{j \neq i}^N \sigma_i \sigma_j \rho_{ij} \frac{\partial^2 V}{\partial x_i^2} + r \sum_{i=1}^N \frac{\partial V}{\partial x_i} \quad (13)$$

where the variance of the random variable Φ_i is

$$\Theta_i = e^{-2x_i} \left| \sum_{j=1}^N \sum_{k=1}^N \frac{\partial^2 V}{\partial x_i \partial x_j} \frac{\partial^2 V}{\partial x_i \partial x_k} \sigma_j \sigma_k \rho_{jk} \right|. \quad (14)$$

Setting

$$F(x, V) = \sum_{i=1}^N \frac{e^{x_i}}{\sqrt{\Delta t}} E \left[C \left(\sqrt{\Delta t} |\Phi_i| \right) |\Phi_i| \right] \quad (15)$$

and

$$\mathcal{L}(V) = \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N \sigma_i \sigma_j \rho_{ij} \frac{\partial^2 V}{\partial x_i \partial x_j} + \sum_{i=1}^N \frac{\partial V}{\partial x_i} \left(r - \frac{\sigma_i^2}{2} \right) - rV, \quad (16)$$

we define the nonlinear PDE for the problem of pricing a multi-asset option with transaction costs as

$$\begin{aligned}-V_\tau + \mathcal{L}V &= FV \quad \text{in } \Omega \times [0, T] \\ V(x_1, \dots, x_n, 0) &= V_0(X) \quad \text{in } \Omega \\ V(x_1, \dots, x_n, \tau) &= g(X, \tau) \quad \text{in } \partial\Omega \times (0, T)\end{aligned} \quad (17)$$

where Ω is a bounded smooth domain in \mathbb{R}^N and $Q_T = \Omega \times (0, T)$, $T > 0$ the parabolic domain. Also functions V_0 and g define the initial condition and boundary condition for the problem such that V_0 belongs to the space $H^2(\overline{Q}_T)$ and g belongs to $H^{2,1}(\partial\Omega \times (0, T))$.

The main theorem of the paper reads as follows.

Theorem 3.1. *Let \mathcal{L} , F , V_0 and g as before. Then problem (17) admits at least one solution.*

Our proof shall be based on an iterative method. With this aim, let us firstly recall the following classical lemma that asserts that the problem with $F := 0$ has a unique solution. The proof follows from Chapter 4 Theorem 9.2 of [10].

Lemma 3.2. *There exists a unique solution $\Lambda \in W_p^{2,1}$ ($p > 1$) to the parabolic problem*

$$\begin{aligned} -V_\tau + \mathcal{L}V &= 0 & \text{in } \Omega \times [0, T] \\ V(x_1, \dots, x_n, 0) &= V_0(X) & \text{in } \Omega \\ V(x_1, \dots, x_n, \tau) &= g(X, \tau) & \text{in } \partial\Omega \times (0, T). \end{aligned}$$

The preceding lemma allows to transform problem (17) into a problem with zero boundary-initial condition by the following change of variables:

$$\begin{aligned} U(X, \tau) &= V(X, \tau) - \Lambda(X, \tau), \\ U_0(X) &= V_0(X) - \Lambda(X, \tau) = 0. \end{aligned}$$

Thus, the original problem becomes

$$\begin{aligned} -U_\tau + \mathcal{L}U &= F(U + \Lambda) & \text{in } \Omega \times [0, T] \\ U(x_1, \dots, x_n, 0) &= 0 & \text{in } \Omega \\ U(x_1, \dots, x_n, \tau) &= 0 & \text{in } \partial\Omega \times (0, T). \end{aligned} \tag{18}$$

It is readily verified that equation (17) has a solution if and only if equation (18) has a solution. We shall search for a weak solution of (18) which, in this context, has the following meaning:

Definition 3.3. U is a solution of (2) in the weak sense if $\forall \phi \in L^2$, we find that

$$\int \int \left(\frac{\partial \phi}{\partial \tau} + \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N \sigma_i \sigma_j \rho_{ij} \frac{\partial^2 \phi}{\partial x_i \partial x_j} - \sum_{i=1}^N \left(r - \frac{1}{2} \sigma_i^2 \right) \frac{\partial \phi}{\partial x_i} - r \phi \right) U \, dx d\tau = \int \int F(U + \Lambda) \phi \, dx d\tau. \tag{19}$$

In order to solve (18), we shall apply an iterative process over the linearised problem. Hence, we consider the linear problem

$$\begin{aligned} -\beta_\tau + \mathcal{L}\beta &= F(\alpha + \Lambda) & \text{in } \Omega \times [0, T] \\ \beta(x_1, \dots, x_n, 0) &= 0 & \text{in } \Omega \\ \beta(x_1, \dots, x_n, \tau) &= 0 & \text{in } \partial\Omega \times (0, T) \end{aligned} \tag{20}$$

with an arbitrary $\alpha \in W_p^{2,1}(\overline{Q}_T)$. Based again on Chapter 4 Theorem 9.2 from [10], there exists a solution $\beta \in W_p^{2,1}(\overline{Q}_T)$ with $p \in (1, \infty)$. This solution can be indeed found by using the integral representation, expressed in terms of its corresponding Green function.

Hence, we propose the following iterative problem:

$$\begin{aligned} -U_\tau^n + \mathcal{L}U^n &= F(U^{n-1} + \Lambda) & \text{in } \Omega \times [0, T] \\ U^n(x_1, \dots, x_n, 0) &= 0 & \text{in } \Omega \\ U^n(x_1, \dots, x_n, \tau) &= 0 & \text{in } \partial\Omega \times (0, T). \end{aligned} \tag{21}$$

with $U^n \in W_p^{2,1}(\overline{Q}_T)$, $n > 1$ and $U^0 = 0$.

3.2 Proof of Theorem 3.1

As mentioned, the solution of the linearised problem (21) has an integral representation given by its Green function. This representation is then used to prove that the sequence of solutions of (21) are uniformly bounded on a predefined subinterval and converges to the solution of (18).

In more precise terms, for $n \geq 1$, the solution U^n can be written as

$$U^n(X, \tau) = \int_0^\tau \int_\Omega G(X, Y, \tau, \tau') F^{n-1} \, dY d\tau' \tag{22}$$

where we denote $F^{n-1} := F(U^{n-1} + \Lambda)$. We shall use this expression to compute the first and second derivatives of U^n with respect to X . For $1 \leq i \leq N$, this yields

$$\begin{aligned} U_{x_i}^n(X, \tau) &= \int_0^\tau \int_\Omega \frac{\partial G}{\partial x_i}(X, Y, \tau, \tau') F^{n-1} dY d\tau' \\ U_{x_i, x_j}^n(X, \tau) &= \int_0^\tau \int_\Omega \frac{\partial G}{\partial x_i \partial x_j}(X, Y, \tau, \tau') F^{n-1} dY d\tau' \end{aligned}$$

Estimates for the Green function and its derivatives can be found in [10]; specifically, Formula 13.1 in Section 13 states that

$$|D_\tau^r D_x^s G(X, Y, \tau, \tau')| \leq c(\tau - \tau')^{-\frac{N+2r+s}{2}} \exp\left(-C \frac{|X - Y|^2}{\tau - \tau'}\right)$$

where $2r + s \leq 2$, $\tau > \tau'$. For our purpose, we take $r = 0$ and $s = 0, 1, 2$. Hence, there exist constants $C_1, C_2 > 0$ such that

$$\begin{aligned} |G(X, Y, \tau, \tau')| &\leq C_1 (\tau - \tau')^{-\frac{N}{2}} \exp\left(-C_2 \frac{\sum_{i=1}^N (x_i - y_i)^2}{\tau - \tau'}\right), \\ \left|\frac{\partial G}{\partial x_i}(X, Y, \tau, \tau')\right| &\leq C_1 (\tau - \tau')^{-\frac{N+1}{2}} \exp\left(-C_2 \frac{\sum_{i=1}^N (x_i - y_i)^2}{\tau - \tau'}\right), \\ \left|\frac{\partial G}{\partial x_i \partial x_j}(X, Y, \tau, \tau')\right| &\leq C_1 (\tau - \tau')^{-\frac{N+2}{2}} \exp\left(-C_2 \frac{\sum_{i=1}^N (x_i - y_i)^2}{\tau - \tau'}\right). \end{aligned} \quad (23)$$

By applying these estimates in (21), we get

$$\begin{aligned} \|U^n(\cdot, \tau)\|_{W_\infty^2} &= \|U^n(\cdot, \tau)\|_{L^\infty} + \sum_{i=1}^N \left\| \frac{\partial U^n}{\partial x_i}(\cdot, \tau) \right\|_{L^\infty} + \sum_{i=1}^N \sum_{j=1}^N \left\| \frac{\partial^2 U^n}{\partial x_i \partial x_j}(\cdot, \tau) \right\|_{L^\infty} \\ &\leq \int_0^\tau \int_\Omega \|G(\cdot, Y, \tau, \tau')\|_{L^\infty} |F^{n-1}| dY d\tau' + \sum_{i=1}^N \int_0^\tau \int_\Omega \left\| \frac{\partial G}{\partial x_i}(\cdot, Y, \tau, \tau') \right\|_{L^\infty} |F^{n-1}| dY d\tau' \\ &\quad + \sum_{i=1}^N \sum_{j=1}^N \int_0^\tau \left\| \int_\Omega \frac{\partial^2 G}{\partial x_i \partial x_j}(\cdot, Y, \tau, \tau') F^{n-1} \right\|_{L^\infty} dY d\tau'. \end{aligned} \quad (24)$$

In order to find a uniform bound for U^n , we shall divide the problem into three different terms which involve the Green function and its spatial derivatives. We recall Lemma 3.4, which gives upper bounds for the nonlinear term F of the PDE and that will help us to find the upper bound of U^n .

Lemma 3.4. *Let F be the nonlinear transaction cost function given by*

$$F(x, U) = \sum_{i=1}^N \frac{e^{x_i}}{\sqrt{\Delta t}} E \left[C \left(\sqrt{\Delta t} |\Phi_i| \right) |\Phi_i| \right]. \quad (25)$$

Then, there exist constants $\bar{K}, C_T > 0$ such that

$$|F(U^{n-1} + \Lambda)| \leq \bar{K} \|U^{n-1}\|_{W_\infty^2} + C_T \quad (26)$$

Proof. Following the definition of F in (25) and taking the absolute value, we get

$$|F^{n-1}| := |F(U^{n-1} + \Lambda)| \leq \sum_{i=1}^N \frac{e^{x_i}}{\sqrt{\Delta t}} \left| E \left[C \left(\sqrt{\Delta t} |\Phi_i| \right) |\Phi_i| \right] \right|. \quad (27)$$

By applying Cauchy-Schwarz inequality on the expected value, we see that

$$\left| E \left[C \left(\sqrt{\Delta t} |\Phi_i| \right) |\Phi_i| \right] \right| \leq \sqrt{E \left[C \left(\sqrt{\Delta t} |\Phi_i| \right)^2 \right]} \sqrt{E \left[|\Phi_i|^2 \right]} \quad (28)$$

For the first term, we know that as $C \in L^2(\Omega)$, the integral is bounded. For the second term, we multiply and divide by Θ_i and considering that $E[|\Phi_i|^2/\Theta_i] = 1$, it can be seen that

$$\sqrt{E[|\Phi_i|^2]} = \sqrt{\Theta_i E\left[\frac{|\Phi_i|^2}{\Theta_i}\right]} = \sqrt{\Theta_i}$$

Therefore,

$$\sqrt{\Theta_i} = e^{-x_i} \sqrt{\left| \sum_{j=1}^N \sum_{k=1}^N \frac{\partial^2 V}{\partial x_i \partial x_j} \frac{\partial^2 V}{\partial x_i \partial x_k} \sigma_j \sigma_k \rho_{jk} \right|} \leq e^{-x_i} \sqrt{\sum_{j=1}^N \sum_{k=1}^N \left| \frac{\partial^2 V}{\partial x_i \partial x_j} \right| \left| \frac{\partial^2 V}{\partial x_i \partial x_k} \right| |\sigma_j| |\sigma_k| |\rho_{jk}|}.$$

As we can consider appropriate bounds for σ_i and ρ_{ij} , there exist $\bar{\sigma}$ such that $\sigma_i \leq \bar{\sigma} \quad \forall 1 \leq i \leq N$ and $|\rho_{ij}| \leq 1 \quad \forall 1 \leq i, j \leq N$. Hence, by rearranging the terms of the second derivatives, it can be seen that

$$\sqrt{\Theta_i} \leq \bar{\sigma}^2 e^{-x_i} \sqrt{\sum_{j=1}^N \sum_{k=1}^N \left| \frac{\partial^2 V}{\partial x_i \partial x_j} \right| \left| \frac{\partial^2 V}{\partial x_i \partial x_k} \right|} = \bar{\sigma}^2 e^{-x_i} \sum_{j=1}^N \left| \frac{\partial^2 V}{\partial x_i \partial x_j} \right| \quad (29)$$

Recalling (27) and applying the results obtained in (28) and (29), it is seen that

$$\begin{aligned} |F(U^{n-1} + \Lambda)| &\leq \sum_{i=1}^N \frac{e^{x_i}}{\sqrt{\Delta t}} \left| E \left[C \left(\sqrt{\Delta t} |\Phi_i| \right) |\Phi_i| \right] \right| \\ &\leq \frac{K \bar{\sigma}^2}{\sqrt{\Delta t}} \sum_{i=1}^N \sum_{j=1}^N \left| \frac{\partial^2 (U^{n-1} + \Lambda)}{\partial x_i \partial x_j} \right| \\ &\leq \bar{K} \|U^{n-1} + \Lambda\|_{W_\infty^2} \\ &\leq \bar{K} \|U^{n-1}\|_{W_\infty^2} + C_T \end{aligned} \quad (30)$$

where C_T comes from the upper estimate of Λ in $[0, T]$. Similar calculation is done in [14]. \square

3.2.1 Estimates for the first term of the inequality

Recalling the inequality (24) we analyse, in the first place, the initial term involving $\|G(\cdot, Y, \tau, \tau')\|_{L^\infty}$. Using the estimates found in (23), it is seen that

$$\begin{aligned} \int_0^\tau \int_\Omega \|G(\cdot, Y, \tau, \tau')\|_{L^\infty} dY d\tau' &\leq \int_0^\tau \int_\Omega C_1 (\tau - \tau')^{-\frac{N}{2}} \exp \left(-C_2 \frac{\sum_{i=1}^N (x_i - y_i)^2}{\tau - \tau'} \right) dY d\tau' \\ &\leq \int_0^\tau \int_{\mathbb{R}^N} C_1 (\tau - \tau')^{-\frac{N}{2}} \exp \left(-C_2 \frac{\sum_{i=1}^N (x_i - y_i)^2}{\tau - \tau'} \right) dY d\tau' \\ &\leq C_1 \left(\frac{\pi}{C_2} \right)^{\frac{N}{2}} \int_0^\tau \int_{\mathbb{R}^N} \pi^{-\frac{N}{2}} \left(\frac{\tau - \tau'}{C_2} \right)^{-\frac{N}{2}} \exp \left(-\frac{\sum_{i=1}^N (x_i - y_i)^2}{\frac{\tau - \tau'}{C_2}} \right) dY d\tau' \\ &\leq \int_0^\tau C_1 \left(\frac{\pi}{C_2} \right)^{\frac{N}{2}} d\tau'. \end{aligned} \quad (31)$$

Hence, by (30) and (31), the first term of (24) has the following upper bound

$$\int_0^\tau \int_\Omega \|G(\cdot, Y, \tau, \tau')\|_{L^\infty} |F^{n-1}| dY d\tau' \leq \int_0^\tau C_1 \left(\frac{\pi}{C_2} \right)^{\frac{N}{2}} \left(\bar{K} \|U^{n-1}\|_{W_\infty^2} + C_T \right) d\tau'. \quad (32)$$

3.2.2 Estimates for the second term of the inequality

Second term of (24) involves analyzing bounds for $\left\| \frac{\partial G}{\partial x_i}(\cdot, Y, \tau, \tau') \right\|_{L^\infty}$. Again, by using the estimates found in (23), we see that

$$\begin{aligned}
\int_0^\tau \int_\Omega \left\| \frac{\partial G}{\partial x_i}(\cdot, Y, \tau, \tau') \right\|_{L^\infty} dY d\tau' &\leq \int_0^\tau \int_\Omega C_1 (\tau - \tau')^{-\frac{N+1}{2}} \exp \left(-C_2 \frac{\sum_{i=1}^N (x_i - y_i)^2}{\tau - \tau'} \right) dY d\tau' \\
&\leq \int_0^\tau \int_{\mathbb{R}^N} C_1 (\tau - \tau')^{-\frac{N+1}{2}} \exp \left(-C_2 \frac{\sum_{i=1}^N (x_i - y_i)^2}{\tau - \tau'} \right) dY d\tau' \\
&\leq \frac{C_1}{(\tau - \tau')^{\frac{1}{2}}} \left(\frac{\pi}{C_2} \right)^{\frac{N}{2}} \int_0^\tau \int_{\mathbb{R}^N} \pi^{-\frac{N}{2}} \left(\frac{\tau - \tau'}{C_2} \right)^{-\frac{N}{2}} \exp \left(-\frac{\sum_{i=1}^N (x_i - y_i)^2}{\frac{\tau - \tau'}{C_2}} \right) dY d\tau' \\
&\leq C_1 \left(\frac{\pi}{C_2} \right)^{\frac{N}{2}} \int_0^\tau (\tau - \tau')^{-\frac{1}{2}} d\tau'.
\end{aligned} \tag{33}$$

Therefore, by using both (30) and (33), we get

$$\int_0^\tau \int_\Omega \left\| \frac{\partial G}{\partial x_i}(\cdot, Y, \tau, \tau') \right\|_{L^\infty} |F^{n-1}| dY d\tau' \leq C_1 \left(\frac{\pi}{C_2} \right)^{\frac{N}{2}} \int_0^\tau (\tau - \tau')^{-\frac{1}{2}} \left(\bar{K} \|U^{n-1}\|_{W_\infty^2} + C_T \right) d\tau'. \tag{34}$$

3.2.3 Estimates for the third term of the inequality

For the third term we recall Lemma 2.1 of [16]. Then, there exists a constant C_3 independent of T and $0 < \beta < 1$ such that

$$\left\| \int_\Omega \frac{\partial^2 G}{\partial x_i \partial x_j}(\cdot, Y, \tau, \tau') dY \right\|_{L^\infty} \leq C_3 (\tau - \tau')^{-\beta}. \tag{35}$$

Then, it follows that

$$\begin{aligned}
&\int_0^\tau \left\| \int_\Omega \frac{\partial^2 G}{\partial x_i \partial x_j}(\cdot, Y, \tau, \tau') F^{n-1}(Y, \tau') dY \right\|_{L^\infty} d\tau' \\
&= \int_0^\tau \left\| \int_\Omega \frac{\partial^2 G}{\partial x_i \partial x_j}(\cdot, Y, \tau, \tau') [F^{n-1}(Y, \tau') - F^{n-1}(X, \tau) + F^{n-1}(X, \tau)] dY \right\|_{L^\infty} d\tau' \\
&\leq \int_0^\tau \left\| \int_\Omega \frac{\partial^2 G}{\partial x_i \partial x_j}(\cdot, Y, \tau, \tau') [F^{n-1}(Y, \tau') - F^{n-1}(X, \tau)] dY \right\|_{L^\infty} + \int_0^\tau \left\| \int_\Omega \frac{\partial^2 G}{\partial x_i \partial x_j}(\cdot, Y, \tau, \tau') F^{n-1}(X, \tau) dY \right\|_{L^\infty} d\tau'
\end{aligned} \tag{36}$$

Using that F^{n-1} belongs to a Holder space for both spatial and time variables, we can find a constant $C_4 > 0$ independent of both spatial and time variables such that

$$|F^{n-1}(Y, \tau') - F^{n-1}(X, \tau)| \leq C_4 \left(\sqrt{\sum_{i=1}^N (x_i - y_i)^2} \right)^\delta.$$

Hence, it can be seen that

$$\begin{aligned}
&\left| \int_\Omega \frac{\partial^2 G}{\partial x_i \partial x_j}(X, Y, \tau, \tau') [F^{n-1}(Y, \tau') - F^{n-1}(X, \tau)] dY \right| \\
&\leq \int_\Omega C_1 (\tau - \tau')^{-\frac{N+2}{2}} \exp \left(-C_2 \frac{\sum_{i=1}^N (x_i - y_i)^2}{\tau - \tau'} \right) C_4 \left(\sum_{i=1}^N (x_i - y_i)^2 \right)^{\frac{\delta}{2}} dY, \\
&\leq \frac{C_1 C_4}{(\tau - \tau')^{\frac{N+2}{2}}} \int_{\mathbb{R}^N} \exp \left(-C_2 \frac{\sum_{i=1}^N (x_i - y_i)^2}{\tau - \tau'} \right) \left(\sum_{i=1}^N (x_i - y_i)^2 \right)^{\frac{\delta}{2}} dY.
\end{aligned} \tag{37}$$

Recalling $\alpha = C_2/(\tau - \tau')$ and $z_i = (y_i - x_i)$, then the integral becomes

$$\int_{\mathbb{R}^N} \exp \left(-C_2 \frac{\sum_{i=1}^N (x_i - y_i)^2}{\tau - \tau'} \right) \left(\sum_{i=1}^N (x_i - y_i)^2 \right)^{\frac{\delta}{2}} dY = \int_{\mathbb{R}^N} \exp \left(-\alpha \sum_{i=1}^N z_i^2 \right) \left(\sum_{i=1}^N z_i^2 \right)^{\frac{\delta}{2}} dZ.$$

In order to solve last integral, we apply a change of variables using the spherical coordinate system for an n-sphere. Hence, the change of variables proposed is obtained by setting

$$\begin{aligned} z_1 &= r \cos \theta_1, \\ z_2 &= r \sin \theta_1 \cos \theta_2, \\ z_3 &= r \sin \theta_1 \sin \theta_2 \cos \theta_3, \\ &\vdots \\ z_{N-1} &= r \sin \theta_1 \dots \sin \theta_{N-2} \cos \theta_{N-1}, \\ z_N &= r \sin \theta_1 \dots \sin \theta_{N-2} \sin \theta_{N-1}, \end{aligned}$$

where $\theta_1, \theta_2, \dots, \theta_{N-2}$ range over $[0, \pi)$ and θ_{N-1} over $[0, 2\pi)$. The Jacobian of this transformation is given by

$$J = r^{N-1} \sin^{N-2} \theta_1 \sin^{N-3} \theta_2 \dots \sin \theta_{N-2} dr d\theta_1 \dots d\theta_{N-1}.$$

Therefore, the integral is given by

$$\begin{aligned} &\int_0^{2\pi} \int_{S_{N-2},0}^\pi \int_0^{+\infty} r^\delta e^{-\alpha r^2} r^{N-1} \sin^{N-2} \theta_1 \sin^{N-3} \theta_2 \dots \sin \theta_{N-2} dr d\theta_1 \dots d\theta_{N-1}, \\ &= 2\pi \left[\int_0^{+\infty} r^{\delta+N-1} e^{-\alpha r^2} dr \right] \left[\int_0^\pi \sin^{N-2} \theta_1 \sin^{N-3} \theta_2 \dots \sin \theta_{N-2} d\theta_1 \dots d\theta_{N-2} \right] \end{aligned}$$

The second integral is nonzero as involves integrating different powers of the sine function between 0 and π . For the first integral term, we can notice that

$$\begin{aligned} \exp \left(-C_2 \frac{r^2}{(\tau - \tau')^2} \right) &= \exp \left(-C_2 \left[\frac{r}{(\tau - \tau')^{1/2}} \right]^2 \right) \\ &= \exp \left(-C_2 \left[\frac{r^{N+\delta}}{(\tau - \tau')^{\frac{N+\delta}{2}}} \right]^{\frac{2}{N+\delta}} \right). \end{aligned}$$

If we make the substitution $p = \frac{r^{N+\delta}}{(\tau - \tau')^{\frac{N+\delta}{2}}}$, then we get that $dp = (N + \delta) r^{N+\delta-1} \frac{1}{(\tau - \tau')^{\frac{N+\delta}{2}}} dr$. Hence,

$$\begin{aligned} \int_0^{+\infty} r^{\delta+N-1} e^{-\alpha r^2} dr &= (\tau - \tau')^{\frac{N+\delta}{2}} (N + \delta) \int_0^{+\infty} e^{-C_2 p^{\frac{2}{N+\delta}}} dp \\ &\leq \tilde{C} (\tau - \tau')^{\frac{N+\delta}{2}}. \end{aligned}$$

Thus, by tracking down (37), we see that

$$\begin{aligned} &\left| \int_\Omega \frac{\partial^2 G}{\partial x_i \partial x_j} (X, Y, \tau, \tau') [F^{n-1}(Y, \tau') - F^{n-1}(X, \tau)] dY \right| \\ &\leq \frac{C_1 C_4}{(\tau - \tau')^{\frac{N+2}{2}}} \int_{\mathbb{R}^N} \exp \left(-C_2 \frac{\sum_{i=1}^N (x_i - y_i)^2}{\tau - \tau'} \right) \left(\sqrt{\sum_{i=1}^N (x_i - y_i)^2} \right)^\delta dY \\ &\leq \frac{\tilde{C}}{(\tau - \tau')^{\frac{N+2}{2}}} (\tau - \tau')^{\frac{N+\delta}{2}} = \tilde{C} (\tau - \tau')^{-1+\frac{\delta}{2}}. \end{aligned}$$

Using (35) and the last result, both terms of (36) are bounded by

$$\begin{aligned} &\int_0^\tau \left\| \int_\Omega \frac{\partial^2 G}{\partial x_i \partial x_j} (\cdot, Y, \tau, \tau') F^{n-1}(Y, \tau') dY \right\|_{L^\infty} \\ &\leq \int_0^\tau \tilde{C} (\tau - \tau')^{-1+\frac{\delta}{2}} d\tau' + \int_0^\tau C_3 (\tau - \tau')^\beta \left(\bar{K} \|U^{n-1}\|_{W_\infty^2} + C_T \right) d\tau'. \end{aligned}$$

By taking $\gamma = \min(\beta, -1 + \frac{\delta}{2})$, we obtain

$$\int_0^\tau \left\| \int_\Omega \frac{\partial^2 G}{\partial x_i \partial x_j}(\cdot, Y, \tau, \tau') F^{n-1}(Y, \tau') dY \right\|_{L^\infty} d\tau' \leq \int_0^\tau \tilde{C}(\tau - \tau')^{-\gamma} \left(\bar{K} \|U^{n-1}\|_{W_\infty^2} + C_T + 1 \right) d\tau'. \quad (38)$$

3.2.4 Estimates for the complete inequality

Using the results from (32), (34) and (38) in (24) and letting $A = C_1 \left(\frac{\pi}{C_2} \right)^{\frac{N}{2}}$, $B = NA$ and $D = \tilde{C}N^2$, we get

$$\begin{aligned} \|U^n(\cdot, \tau)\|_{W_\infty^2} &\leq \int_0^\tau \left[A + B(\tau - \tau')^{-\frac{1}{2}} + D(\tau - \tau')^{-\gamma} \right] \left(\bar{K} \|U^{n-1}\|_{W_\infty^2} + C_T \right) d\tau' \\ &= C_T \left(A\tau + 2B\tau^{\frac{1}{2}} + D\frac{\tau^{1-\gamma}}{1-\gamma} \right) + \bar{K} \int_0^\tau \left[A + B(\tau - \tau')^{-\frac{1}{2}} + D(\tau - \tau')^{-\gamma} \right] \|U^{n-1}\|_{W_\infty^2} d\tau' \\ &\leq C(T, \gamma) + \bar{K} \int_0^\tau \left[A + B(\tau - \tau')^{-\frac{1}{2}} + D(\tau - \tau')^{-\gamma} \right] \|U^{n-1}\|_{W_\infty^2} d\tau'. \end{aligned} \quad (39)$$

Given the bound obtained in (39), Lemma 3.5 provides the specific subinterval $[0, \tilde{\tau}]$ in which U^n is uniformly bounded.

Lemma 3.5. *There exists $\tilde{\tau} \in [0, T]$ such that $\|U^n(\cdot, \tau)\|_{W_\infty^2}$ is uniformly bounded in $[0, \tilde{\tau}]$.*

Proof. We want to find $\tilde{\tau}$ such that

$$\left| \bar{K} \int_0^{\tilde{\tau}} \left[A + B(\tau - \tau')^{-\frac{1}{2}} + D(\tau - \tau')^{-\gamma} \right] d\tau' \right| < 1.$$

By integrating last equation, we get that

$$\bar{K} \int_0^{\tilde{\tau}} \left[A + B(\tau - \tau')^{-\frac{1}{2}} + D(\tau - \tau')^{-\gamma} \right] d\tau' = \bar{K} \left(A\tilde{\tau} + 2B\tilde{\tau}^{\frac{1}{2}} + D\frac{\tilde{\tau}^{1-\gamma}}{1-\gamma} \right).$$

It is important to notice about two facts. First, that all the constants that are multiplying $\tilde{\tau}$, are positive. Second, that the powers of $\tilde{\tau}$ are all negative. Hence, it is possible to find $\tilde{\tau}$ such that

$$\left| \bar{K} \left(A\tilde{\tau} + 2B\tilde{\tau}^{\frac{1}{2}} + D\frac{\tilde{\tau}^{1-\gamma}}{1-\gamma} \right) \right| < 1.$$

For example, if we set $H = \max\{A\tilde{\tau}\bar{K}, 2B\sqrt{\tilde{\tau}\bar{K}}, \frac{D\bar{K}\tilde{\tau}^{1-\gamma}}{1-\gamma}\}$ and $\zeta = \min\{-1, -1/2, -1 + \gamma\}$, we see that

$$\left| \tilde{\tau}^{-1} (A\bar{K}) + \tilde{\tau}^{-1/2} (2B\bar{K}) + \tilde{\tau}^{-1+\gamma} \left(\frac{D\bar{K}}{1-\gamma} \right) \right| \leq 3H k^\zeta$$

Hence, we can choose $\tilde{\tau}$ such that

$$\tilde{\tau} > (3H)^{-\zeta}.$$

□

Observe also that, as all the previous bounds A, B, D are uniform, we can find a proper integer k and take $\tilde{\tau} = T/k$ in order to apply the same procedure on each sub-interval $[\tilde{\tau}, 2\tilde{\tau}]$, $[2\tilde{\tau}, 3\tilde{\tau}]$ and so on. Therefore, we can split the interval $[0, T]$ in k sub-intervals and solve the differential equation on each of them. So, at first, we solve the equation on $[0, T/k]$. We will find a solution V such that its value at time T/k will be the initial value of the same problem on the next interval. Thus, we present the method to find a solution on the interval $[0, T/k]$.

By recalling (39) and letting

$$R := \int_0^{\tilde{\tau}} \left[A + B(\tau - \tau')^{-\frac{1}{2}} + D(\tau - \tau')^{-\gamma} \right] d\tau',$$

we can see that

$$\begin{aligned}
\|U^n(\cdot, \tau)\|_{W_\infty^2} &\leq C(T, \gamma) + \bar{K} \int_0^\tau \left[A + B(\tau - \tau')^{-\frac{1}{2}} + D(\tau - \tau')^{-\gamma} \right] \|U^{n-1}\|_{W_\infty^2} d\tau' \\
&\leq C(T, \gamma) \left(1 + \bar{K}R + \dots + \bar{K}^{n-1}R^{n-1} \right) \\
&\leq \frac{C(T, \gamma)}{1 - \bar{K}R}
\end{aligned}$$

since $|\bar{K}R| < 1$. Therefore, $\|U^n(\cdot, \tau)\|_{W_\infty^2}$ is uniformly bounded on $[0, \tilde{\tau}]$. Also, using this result with (18), we can see that $\|U_\tau^n(\cdot, \tau)\|_{L_\infty}$ is also uniformly bounded in $[0, \tilde{\tau}]$. Indeed, $U^n(\cdot, \tau)$ is bounded in the $W_\infty^{2,1}$ sense.

Using the Sobolev embedding theorem for $p = \infty$, we have that $W_\infty^{2,1} \hookrightarrow C^{1,0}$. Hence, passing to a subsequence, we have that

$$U^{n_k}(\cdot, \tau) \rightarrow U(\cdot, \tau) \in C^{1,0}(Q_{\tilde{\tau}}), \quad (40)$$

where this convergence is uniform. Also, as $W_\infty^{2,1} \hookrightarrow W_2^{2,1}$, we have that $U^n(\cdot, \tau)$ converges weakly on $H^{2,1}$. Thus, we have to check that U is effectively a solution of (18) in the weak sense. Using the definition of weak solution stated on Definition 3.3, we find that

$$\begin{aligned}
\int \int U \left(\frac{\partial \phi}{\partial \tau} + \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N \sigma_i \sigma_j \rho_{ij} \frac{\partial^2 \phi}{\partial x_i \partial x_j} - \sum_{i=1}^N \left(r - \frac{1}{2} \sigma_i^2 \right) \frac{\partial \phi}{\partial x_i} - r \phi \right) dx d\tau \\
= \int \int \left(-\frac{\partial U}{\partial \tau} + \mathcal{L}U \right) \phi dx d\tau \quad (41)
\end{aligned}$$

$$= \lim_{n \rightarrow +\infty} \int \int \left(-\frac{\partial U^n}{\partial \tau} + \mathcal{L}U^n \right) \phi dx d\tau \quad (42)$$

$$= \lim_{n \rightarrow +\infty} \int \int F(U^n + \Lambda) \phi dx d\tau \quad (43)$$

$$= \int \int F(U + \Lambda) \phi dx d\tau, \quad (44)$$

using integration by parts in (41), the weak convergence in (42) and dominated convergence in (44). Nonetheless, $U \in W_\infty^{2,1}$ because $U \in H^{2,1}$, U is a weak solution and $F \in L^\infty$. If we apply this procedure on each subinterval $[0, T/k]$ with a predefined k , we finally find a solution $U \in W_\infty^{2,1}(Q_T)$ on the whole interval $[0, T]$.

4 Numerical Implementation

4.1 Numerical Framework

On this section we develop a numerical scheme in order to solve the iterative problem defined on (3). Also, we adapt the framework to consider the partition of the interval $[0, T]$ obtained Lemma 3.5. Hence, we recall the linear problem

$$\begin{aligned}
-U_\tau^n + \mathcal{L}U^n &= F(U^{n-1} + \Lambda) \quad \text{in } \Omega \times [0, T] \\
U^n(x_1, \dots, x_n, 0) &= 0 \quad \text{in } \Omega \\
U^n(x_1, \dots, x_n, \tau) &= 0 \quad \text{in } \partial\Omega \times (0, T)
\end{aligned} \quad (45)$$

with $U^0 = 0$ and $\dim \Omega = 2$. For numerical convenience, we approximate the original smooth domain by a discrete one $\hat{\Omega}_T \subset [a, b] \times [a, b] \times [0, T]$, setting a and b in order to cover a set of feasible logarithmic stock prices. The step of the spatial variables is uniformly set as $\Delta x = (b - a)/S_x$, being S_x the number of grid points in the x -direction. The step of the temporal variable is also uniformly set as $\Delta \tau = T/T_x$ being T_x the number of grid points in the τ -direction. We define n to be the step of the iterative problem and, on each n , m to be each of the temporal steps. Hence, we define the solution to the n -step iterative problem as $U_{ij}^m = U(x_i, y_j, m\Delta\tau)$ where $0 \leq i, j \leq S_x$ and $0 \leq m \leq T_x$.

On each step n , we have to solve a linear problem involving both second and mixed derivatives of U . If we apply directly a finite difference scheme, the invertible matrix would not be tridiagonal as mixed spatial

derivatives have to be taken into account. Hence, we apply an Alternating Direction Implicit (ADI) method with a Finite Difference approach (FD).

We follow [9] to determine the two stages of the procedure. The main idea of the ADI method is to generate an intermediate step $n + 1/2$ between steps n and $n + 1$. The first half step is taken implicitly in the x-direction and explicitly in the y-direction. The other half step is taken implicitly in the y-direction and explicitly in the x-direction.

In the first place, we split the temporal derivative as shown on (46)

$$U_\tau = \frac{U_{ij}^{n+1} - U_{ij}^n}{\Delta t} = \frac{U_{ij}^{n+1} - U_{ij}^{n+\frac{1}{2}}}{\Delta t} + \frac{U_{ij}^{n+\frac{1}{2}} - U_{ij}^n}{\Delta t}. \quad (46)$$

Then, we discretize the bilinear operator

$$\mathcal{L} = \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N \sigma_i \sigma_j \rho_{ij} \frac{\partial^2 U}{\partial x_i \partial x_j} + \sum_{i=1}^N \frac{\partial U}{\partial x_i} \left(r - \frac{\sigma_i^2}{2} \right) - rU,$$

setting

$$\begin{aligned} \frac{\partial U}{\partial x_1} &= \frac{U_{i+1,j}^n - U_{i,j}^n}{\Delta x}, \\ \frac{\partial U}{\partial x_2} &= \frac{U_{i,j+1}^n - U_{i,j}^n}{\Delta x}, \\ \frac{\partial^2 U}{\partial x_1^2} &= \frac{U_{i+1,j}^n - 2U_{i,j}^n + U_{i-1,j}^n}{\Delta x^2}, \\ \frac{\partial^2 U}{\partial x_2^2} &= \frac{U_{i,j+1}^n - 2U_{i,j}^n + U_{i,j-1}^n}{\Delta x^2}, \\ \frac{\partial^2 U}{\partial x_1 \partial x_2} &= \frac{U_{i+1,j+1}^n + U_{i-1,j-1}^n - U_{i-1,j}^n - U_{i,j-1}^n}{4\Delta x^2}. \end{aligned}$$

As in section 2.1 of [9], we split the discretization of the operator \mathcal{L} between

$$\begin{aligned} \mathcal{L}^x &= \frac{\sigma_1^2}{4} \frac{U_{i+1,j}^{n+\frac{1}{2}} - 2U_{i,j}^{n+\frac{1}{2}} + U_{i-1,j}^{n+\frac{1}{2}}}{\Delta x^2} + \frac{\sigma_2^2}{4} \frac{U_{i,j+1}^n - 2U_{i,j}^n + U_{i,j-1}^n}{\Delta x^2} + \frac{1}{2} \sigma_1 \sigma_2 \rho \frac{U_{i+1,j+1}^n + U_{i-1,j-1}^n - U_{i-1,j}^n - U_{i,j-1}^n}{4\Delta x^2} \\ &+ \frac{1}{2} \left(r - \frac{\sigma_1^2}{2} \right) \frac{U_{i+1,j}^{n+\frac{1}{2}} - U_{i,j}^{n+\frac{1}{2}}}{\Delta x} + \frac{1}{2} \left(r - \frac{\sigma_2^2}{2} \right) \frac{U_{i,j+1}^n - U_{i,j}^n}{\Delta x} - \frac{1}{2} r U_{ij}^{n+\frac{1}{2}} \end{aligned}$$

and

$$\begin{aligned} \mathcal{L}^y &= \frac{\sigma_1^2}{4} \frac{U_{i+1,j}^{n+\frac{1}{2}} - 2U_{i,j}^{n+\frac{1}{2}} + U_{i-1,j}^{n+\frac{1}{2}}}{\Delta x^2} + \frac{\sigma_2^2}{4} \frac{U_{i,j+1}^{n+1} - 2U_{i,j}^{n+1} + U_{i,j-1}^{n+1}}{\Delta x^2} + \frac{1}{2} \sigma_1 \sigma_2 \rho \frac{U_{i+1,j+1}^{n+\frac{1}{2}} + U_{i-1,j-1}^{n+\frac{1}{2}} - U_{i-1,j}^{n+\frac{1}{2}} - U_{i,j-1}^{n+\frac{1}{2}}}{4\Delta x^2} \\ &+ \frac{1}{2} \left(r - \frac{\sigma_1^2}{2} \right) \frac{U_{i+1,j}^{n+\frac{1}{2}} - U_{i,j}^{n+\frac{1}{2}}}{\Delta x} + \frac{1}{2} \left(r - \frac{\sigma_2^2}{2} \right) \frac{U_{i,j+1}^{n+1} - U_{i,j}^{n+1}}{\Delta x} - \frac{1}{2} r U_{ij}^{n+1} \end{aligned}$$

obtaining a two-stage full scheme

$$\begin{aligned} \frac{U_{ij}^{n+\frac{1}{2}} - U_{ij}^n}{\Delta t} &= \mathcal{L}^x U_{ij}^{n+\frac{1}{2}}, \\ \frac{U_{ij}^{n+1} - U_{ij}^{n+\frac{1}{2}}}{\Delta t} &= \mathcal{L}^y U_{ij}^{n+1}. \end{aligned}$$

As the problem (45) contains the linear function F (which depends only on t and x), we decide to add this term on the second stage of the procedure by redefining $\tilde{\mathcal{L}}^y = \mathcal{L}^y - F$.

$$\frac{U_{ij}^{n+1} - U_{ij}^n}{\Delta t} = \mathcal{L}^x U_{ij}^{n+\frac{1}{2}} + \mathcal{L}^y U_{ij}^{n+1} - F(\cdot) = \mathcal{L}^x U_{ij}^{n+\frac{1}{2}} + \tilde{\mathcal{L}}^y U_{ij}^{n+1}.$$

The proposed framework is used to calculate first $U^{n+\frac{1}{2}}$ and then U^{n+1} . The most important gain with the ADI method is that only requires the solution of two tridiagonal sets of equations at each time step. Further results regarding the convergence, stability and consistency of the numerical method can be found in [7] and [8].

4.2 Numerical Results

4.2.1 Preliminaries

In order to implement the framework proposed in section 3, we selected a type of multi-asset option and a transaction cost function. First, we choose to price a best cash-or-nothing option call on two assets. This option pays out a predefined cash amount K if assets S_1 or S_2 are above or equal to the strike price X . The closed-form formula is presented on [5] as

$$\begin{aligned} c_{best} &= K e^{-rT} [M(y, z_1; -\rho_1) + M(-y, z_2; -\rho_2)] \\ y &= \frac{\ln(S_1/S_2) + \frac{\sigma_2^2 T}{2}}{\sigma \sqrt{T}}, \quad \sigma = \sqrt{\sigma_1^2 + \sigma_2^2 - 2\sigma_1\sigma_2\rho} \\ z_1 &= \frac{\ln(S_1/X) + \frac{\sigma_1^2 T}{2}}{\sigma_1 \sqrt{T}}, \quad z_2 = \frac{\ln(S_2/X) + \frac{\sigma_2^2 T}{2}}{\sigma_2 \sqrt{T}} \\ \rho_1 &= \frac{\sigma_1 - \rho}{\sigma}, \quad \rho_2 = \frac{\sigma_2 - \rho}{\sigma} \end{aligned} \tag{47}$$

where S_1 and S_2 are the stock prices, σ_1 and σ_2 are the volatilities, ρ is the correlation between both assets, T is the maturity and $M(a, b; \rho)$ is

$$M(a, b; \rho) = \frac{1}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^a \int_{-\infty}^b \exp\left[-\frac{x^2 + y^2 - 2\rho xy}{2(1-\rho^2)}\right] dx dy.$$

Second, given the nonlinear function F defined on (15), we choose an exponential decreasing transaction cost function defined as

$$C(x) = C_0 e^{-kx}$$

for each asset x . Hence, by recalling (11), we can see that

$$\begin{aligned} E\left[C\left(\sqrt{\Delta t} |\Phi_i|\right) |\Phi_i|\right] &= \int_0^{+\infty} C_0 e^{-k\sqrt{\Delta t}x} \frac{2x}{\sqrt{2\pi\Theta_i}} e^{-x^2/2\Theta_i} dx \\ &= C_0 \sqrt{\frac{2}{\pi}} \int_0^{+\infty} e^{-k\sqrt{\Delta t}x} \frac{x}{\sqrt{\Theta_i}} e^{-x^2/2\Theta_i} dx \\ &= C_0 \sqrt{\frac{2}{\pi}} \int_0^{+\infty} e^{-k\sqrt{\Delta t\Theta_i}y} \sqrt{\Theta_i}y e^{-y^2/2} dy \\ &= C_0 \sqrt{\Theta_i} \sqrt{\frac{2}{\pi}} \left[1 - e^{k^2\Delta t\Theta_i/2} k\sqrt{\Delta t\Theta_i} \text{ERFC}\left(k\sqrt{\frac{\Delta t\Theta_i}{2}}\right)\right]. \end{aligned}$$

Then,

$$F(t, x) = C_0 \sqrt{\frac{2}{\pi}} \sum_{i=1}^2 \frac{e^{x_i}}{\Delta t} \sqrt{\Theta_i} \left[1 - e^{k^2\Delta t\Theta_i/2} k\sqrt{\Delta t\Theta_i} \text{ERFC}\left(k\sqrt{\frac{\Delta t\Theta_i}{2}}\right)\right]$$

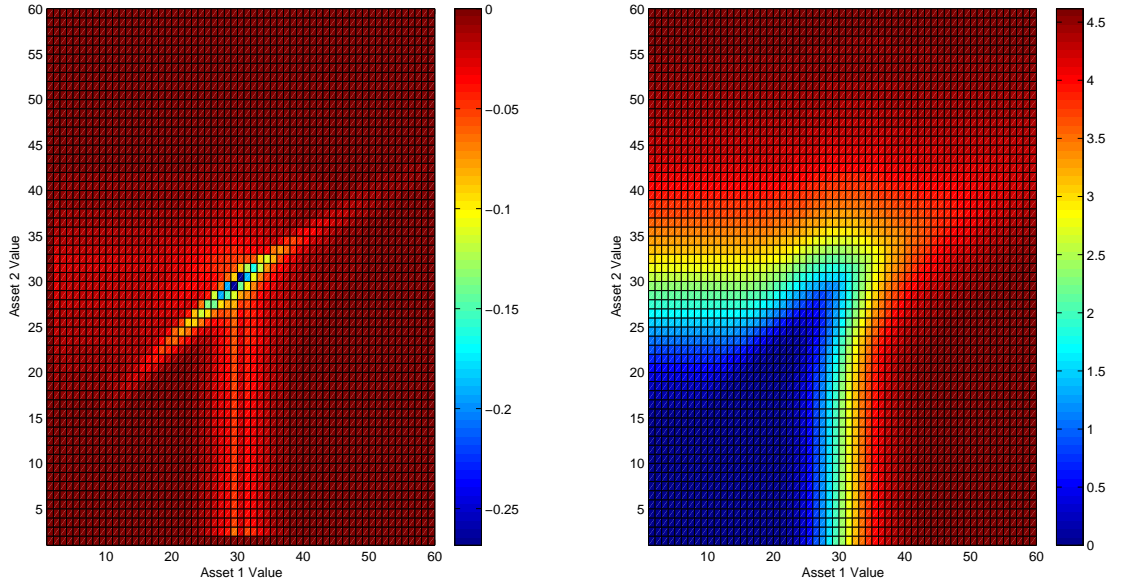
4.2.2 Results

On Table 1 we present the parameters chosen for the numerical implementation. Three different tests were then applied by varying the values of the stocks price, volatility, interest rate and strike among others. Figures 1 to 3 show the results obtained for these three scenarios. Within each figure, two main plots are provided. For example, on Figure 1a, it is shown the solution for (45) at time $\tau = T$. By recalling the original problem, Figure 1b presents the solution for the original problem (17) at time $\tau = T$. The latter would be considered the value of the option price at present time for each possible stock price.

Figures 4 to 6 allow to see how the iterative method works on each scenario fixing specific stocks values. Vertical lines are place at the end of each step in order to confirm that the iterative methodology is converging to a smooth solution.

Parameters	Testing 1		Testing 2		Testing 3	
	Asset 1	Asset 2	Asset 1	Asset 2	Asset 1	Asset 2
σ	0.30	0.15	0.05	0.7	0.2	0.2
ρ		0.5		-0.3		0.2
r		0.08		0.02		0.1
T		1 year		1 year		1 year
K		5		8		6
X		30		40		15
Δx		1		1		1
Δt_{TC}		1/261		1/261		1/261
C_0		0.005		0.001		0.003
k		1		0.5		0.7
Steps		10		10		10

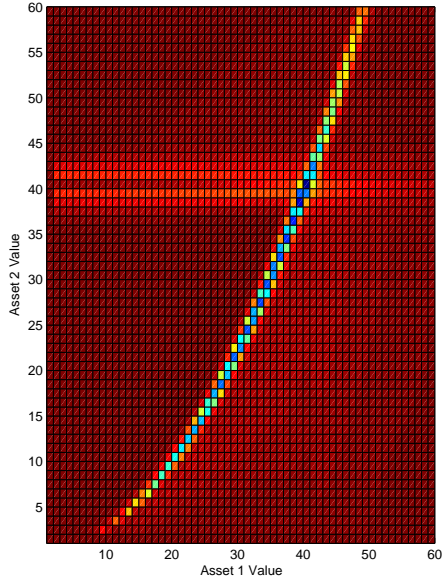
Table 1: Numerical implementation parameters



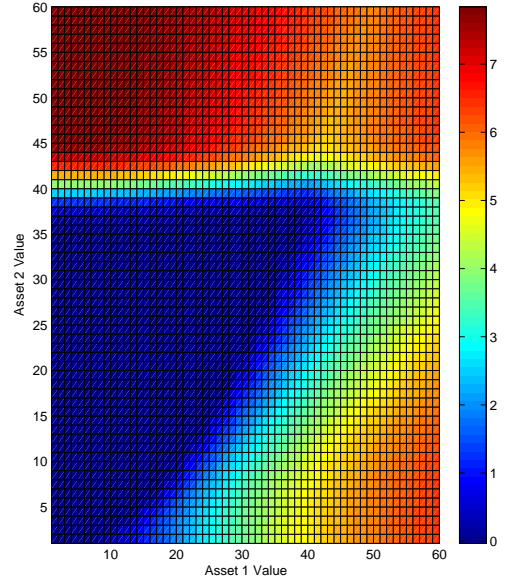
(a) Result for (45) at $\tau = T$: Testing 1

(b) Result for (17) at $\tau = T$: Testing 1

Figure 1: Numerical Results 1

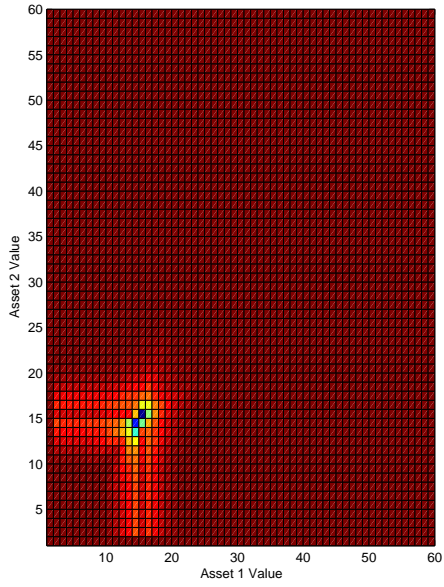


(a) Result for (45) at $\tau = T$: Testing 2

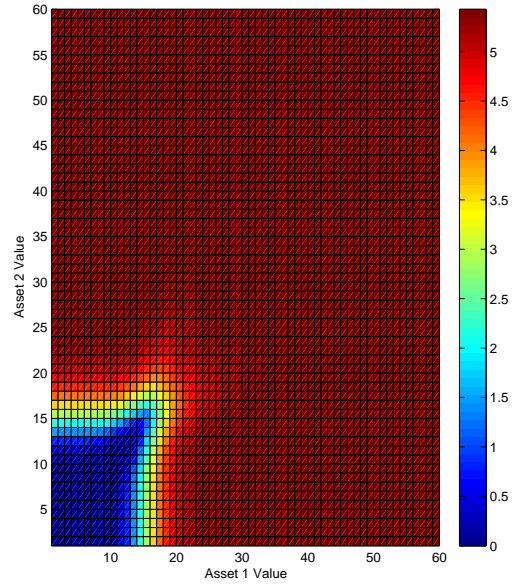


(b) Result for (17) at $\tau = T$: Testing 2

Figure 2: Numerical Results 2

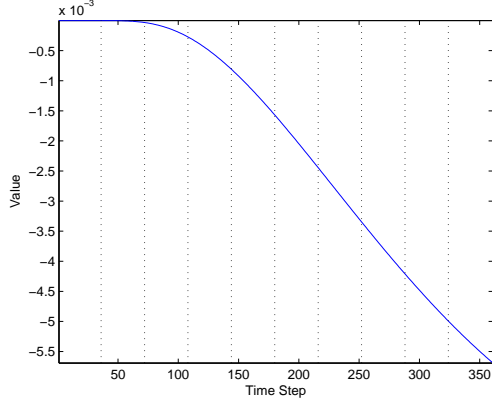


(a) Result of (45) at $\tau = T$: Testing 3

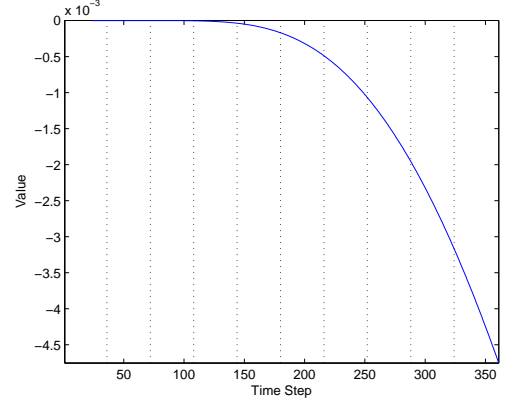


(b) Result for (17) at $\tau = T$: Testing 3

Figure 3: Numerical Results 3

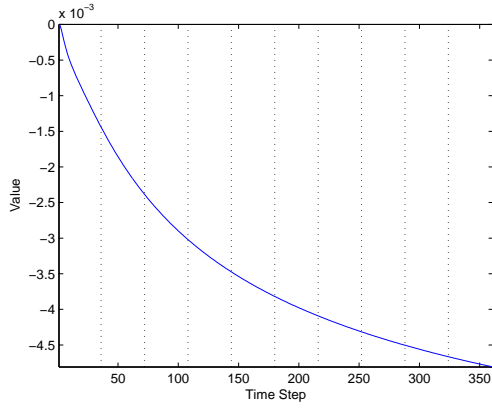


(a) Testing 1: $S_1 = 30, S_2 = 45$

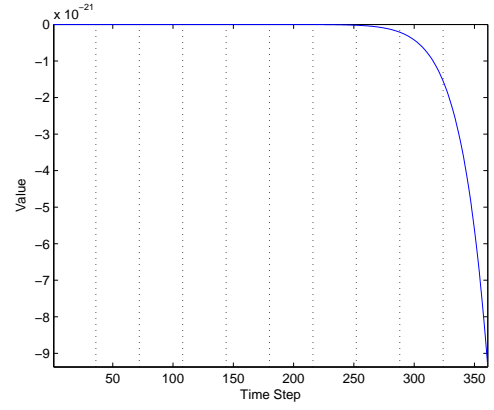


(b) Testing 1: $S_1 = 15, S_2 = 10$

Figure 4: Numerical Results 4

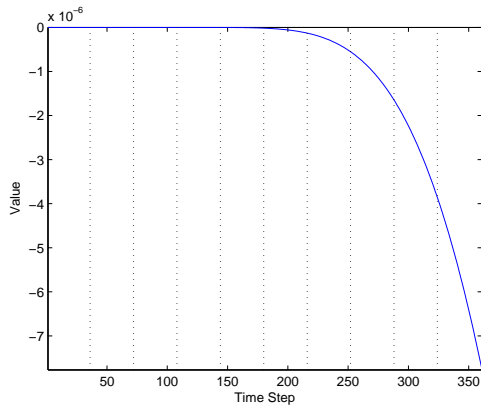


(a) Testing 2: $S_1 = 30, S_2 = 45$

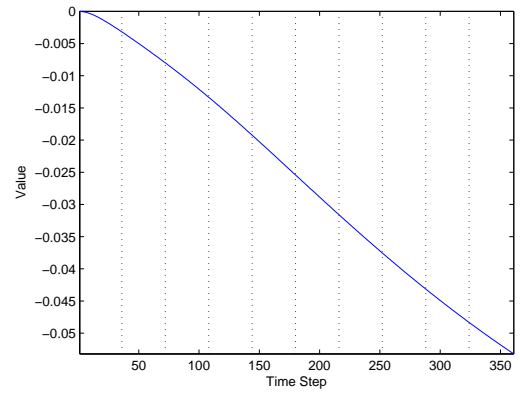


(b) Testing 2: $S_1 = 15, S_2 = 10$

Figure 5: Numerical Results 5



(a) Testing 3: $S_1 = 30, S_2 = 45$



(b) Testing 3: $S_1 = 15, S_2 = 10$

Figure 6: Numerical Results 6

5 Acknowledgement

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